

THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS
MATH3070 (Second Term, 2016–2017)
Introduction to Topology
Exercise 8 Quotient

Remarks

Many of these exercises are adopted from the textbooks (Davis or Munkres). You are suggested to work more from the textbooks or other relevant books.

1. Let $X = \{(x, 0) : x \in \mathbb{R}\} \cup \{(x, 1) : x \in \mathbb{R}\} \subset \mathbb{R}^2$, i.e., $X = \mathbb{R} \amalg \mathbb{R}$. Define an equivalence relation on X by identifying $(0, 0)$ and $(0, 1)$. Rigorously, this means $(s_1, t_1) \sim (s_2, t_2)$ iff $(s_1, t_1) = (s_2, t_2)$ or $(s_1, t_1) = (0, 0)$ while $(s_2, t_2) = (0, 1)$ or vice versa. Show that X/\sim is homeomorphic to the two axes in \mathbb{R}^2 .
2. Let $X = \{(s, t) \in \mathbb{R}^2 : 0 \neq t \in \mathbb{Z}\}$ and $Y = \{(s, t) \in \mathbb{R}^2 : 1/t \in \mathbb{Z}\}$ be given the standard induced topology. Define an equivalence relation on both X and Y by $(s_1, t_1) \sim (s_2, t_2)$ iff $(s_1, t_1) = (s_2, t_2)$ or $s_1 = s_2 = 0$. That is points on the y -axis are identified to one point. Is it true that X/\sim and Y/\sim are homeomorphic?
3. Define an equivalence relation on \mathbb{R} by identifying n with $1/n$ for all $n \in \mathbb{Z}$.
 - (a) Sketch a picture to represent the space \mathbb{R}/\sim .
 - (b) Find a sequence $x_n \in \mathbb{R}$ such that $[x_n] \in \mathbb{R}/\sim$ converges but x_n does not.
 - (c) Can a sequence $x_n \in \mathbb{R}$ converge but $[x_n] \in \mathbb{R}/\sim$ does not?
4. Let X/\sim be a quotient space obtained from X and $Y \subset X$.
 - (a) Show that there is a natural way to induce an equivalence relation on Y ; and thus a quotient space Y/\sim .
 - (b) Let $Y^* = \{[x] \in (X/\sim) : [x] \cap Y \neq \emptyset\}$ be given the topology induced from X/\sim . Is Y^* homeomorphic to Y/\sim ?
5. Let $X = \mathbb{R}/\sim$ where $s \sim t$ if $s, t \in \mathbb{Z}$; also let

$$Y = \bigcup_{n=1}^{\infty} \left\{ z \in \mathbb{C} : \left| z - \frac{1}{n} \right| = \frac{1}{n} \right\} \quad \text{and} \quad Z = \bigcup_{n=1}^{\infty} \{ z \in \mathbb{C} : |z - n| = n \}$$
 with induced topology of $\mathbb{C} = \mathbb{R}^2$. Are X, Y, Z homeomorphic to each other?
6. Let $A \subset X$. What can you say about $\text{Int}(A)/\sim$ and $\text{Int}(A/\sim)$; $\text{Cl}(A)/\sim$ and $\text{Cl}(A/\sim)$?
7. Let \mathfrak{T}_q be the quotient topology on X/\sim and \mathfrak{T}' be any topology. Show that if the quotient map $q: (X, \mathfrak{T}) \rightarrow (X/\sim, \mathfrak{T}')$ is continuous, then $\mathfrak{T}' \subset \mathfrak{T}_q$.

8. Let Z be any topological space. A mapping $f: (X/\sim, \mathfrak{T}_q) \rightarrow Z$ is continuous if and only if $f \circ q: (X, \mathfrak{T}) \rightarrow Z$ is continuous.

If \mathfrak{T}' is a topology on X/\sim satisfying the same property, then $\mathfrak{T}' = \mathfrak{T}_q$.

9. On the disk $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$ with induced topology from standard \mathbb{C} , define an equivalence relation \sim by $z_1 \sim z_2$ if $z_1 = z_2$ or $\bar{z}_1 = z_2$ with $|z_1| = 1 = |z_2|$. Show that \mathbb{D}/\sim is homeomorphic to the sphere \mathbb{S}^2 .

10. Define a mapping $p: \mathbb{D} \rightarrow \mathbb{S}^2$ in the following way. For $z = |z|e^{i\theta} \in \mathbb{D}$, let

$$p(z) = (\sin(\pi|z|)\cos\theta, \sin(\pi|z|)\sin\theta, -\cos(\pi|z|)).$$

(a) Show that the quotient topology on \mathbb{S}^2 induced by p is the standard topology.

(b) Define an equivalence relation \sim on \mathbb{D} such that $z_1 \sim z_2$ if $p(z_1) = p(z_2)$. What is the topological space \mathbb{D}/\sim ?